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FINITE-ELEMENT TECHNIQUE FOR SOLVING PROBLEMS FORMULATED BY HAMILTON'S PRINCIPLE

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Abstract—The finite element technique is applied to functionals which govern dynamical problems, where time is an independent variable.

The present paper demonstrates improved accuracy in mass-spring-damper systems and exemplifies "rendevous problems" of a "travelling" particle in a medium. The motion is governed by Hamilton's principle. The time interval is fixed.

Functionals are constructed from Hamilton's extended principle and appropriate conditions stating the various constraints.

Initial value problems may be incorporated by writing a functional in accordance with Gurtin's method.

Shape functions are polynomials in time and can be extended to other spatial variables when present.

As a result of a variation to the functional, a system of algebraic (not necessarily linear) equations is formed. This system is solved simultaneously to yield its motion within the boundaries of the given time interval.

NOMENCLATURE

Abbreviations

- CFS closed form solution
- HPFE Hamiliton's principle with finite elements
- RKS Runge-Kutta's solution
- SFFE second functional with finite elements
- VP velocity's power

Symbols

- $(^\circ) \equiv d/dt$ time derivation
- ( ) vector
- $( )^e$ elements independent parameters at the nodes
- $( )_E$ that belongs to an element
- j,k element's nodes
- $*$ convolution product
- $\delta( )$ variational operator
- Δ time step
- B damping coefficient
- f nonconservative external force
- $\hat{F}$ impulse
- g gravity acceleration
- k spring coefficient
- L the Lagrangian; $L = T - V$
- m mass
- N shape function
- $\bar{n}$ power of particle's velocity when introduced in Hamilton's principle; $\bar{n} = 1/2$ (VP-1)
- q generalized coordinate
- $\dot{S}$ particle's velocity
- t time
- t_0, t_f time limits at the origin and final time interval
- T system's kinetic energy
- V system's potential energy
- x,y coordinates

Greek symbols

- J functional
- τ normalized time
- ω natural frequency
- ω_1 specific damping coefficient; $\omega_1 = B/m$

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INTRODUCTION

In a finite element analysis for time dependent phenomena, a variation is usually applied to the spatial approximations of the dependent variables [1–3], yielding a system of ordinary differential equations in time which yield an initial value problem. Solutions to such systems are well documented.

Fried[4] used Hamilton's principle to develop a procedure which can be regarded as a generalization of the step-by-step method for integrating the dynamic equation of motion. Vance and Sitchin[5] received a system of recurrence equations using the finite differences method, and by imposing Lagrange multipliers into Hamilton's principle.

Papers dealing with the question of whether Hamilton's principle is an extremum principle, were presented by Leitmann[6], Smith, D. R. and Smith, C. V.[7]. Gurtin in his papers[8, 9] extended the application of variational method by showing how initial conditions may be incorporated explicitly in the variational formulation of a problem, the limitation of his method being that only linear problems can be dealt with. Sandhu and Wilson[10] had applied the finite element method to functionals of Gurtin's method for fluid flow in porous media.

The present method includes the time variable in the functional to be varied. Boundary conditions at both ends of the time interval are given. The entire time interval (the only independent variable at the present work) is also a given parameter.

Applying the finite element technique to a given functional, a differential formulation, that of searching an extremum to a function of several variables is obtained.

In contrast to initial value problems techniques, here a system of algebraic equations is derived from Hamilton's extended principle. The system is solved simultaneously. This fact prevents an accumulation of round-off errors.

Various constraints, when present, are imposed as appropriate additions to corresponding equations of the system.

An example showing an application of the present method to a Gurtin type functional will be shown later on.

For some examples that were illustrated, a comparison was made between: (1) the closed form solution; (2) a solution based on the Runge–Kutta method; (3) a solution derived from a functional not identical to Hamilton's extended principle.

For the problems that were solved it was demonstrated that applying the finite element method to Hamilton's principle (being a stationary principle) gives better correlation to the closed form solution than using either the Runge–Kutta method, or the solution which was generated in Step (3) above.

The work done here is based on the M.Sc. research thesis of the first author[11] which was performed during the year 1975.

DYNAMICAL SYSTEMS

Hamilton's extended principle states that[5]:

$$\delta J = \delta \int_{t_0}^{t_f} L\,\mathrm{d}t + \int_{t_0}^{t_f} \Big(\sum_{i=1} f_i \delta q_i\Big)\mathrm{d}t = 0 \tag{1}$$

noting that the variation is not applied to non-conservative external forces [12]. Equation (1), due to the fact that not all of its parts underwent a variation, is not an extremum of J. Adding the expression, $\int(\Sigma_i f_i \delta q_i)\mathrm{d}t$, Hamilton's extended principle may be regarded as a virtual work principle.

Leitmann[6] presented for certain types of field differential equations, functionals such that when variation is applied to all of their expressions the field equations result. A finite element desecretization for approximating generalized coordinates in the element would be:

$$q_E \cong \hat{q}_E = \mathrm{N}_E{}^T(t)\mathbf{q}_E{}^e. \tag{2}$$

Two degrees of freedom at each element's node yields:

$$\mathbf{q}_E{}^e = [q_j, q_{j+1}, \mathring{q}_j, \mathring{q}_{j+1}]^T. \tag{3}$$

Normalizing shape functions so that each one equals unity at the correspondent element's node and zero at the second node, yields third degree polynomials:

$$\begin{aligned} N_E &= [N_1, N_2, N_3, N_3, N_4]^T \\ &= [(1-3\tau^2+2\tau^3), (3\tau^2-2\tau^3), \\ &\quad \Delta_E\tau(1-\tau)^2, -\Delta_E\tau^2(1-\tau)]^T \end{aligned} \tag{4}$$

$$\Delta_E = t_{j+1} - t_j;\ \tau = \frac{t-t_j}{\Delta_E}.$$

Inserting the finite element desecretization into the functional yields a function with several independent variables, as many as the number of degrees of freedom which are imposed on element's nodes. For two degrees of freedom, we have:

$$J = J(q_k, \mathring{q}_k, t). \tag{5}$$

Proceeding with the variation, a formulation that of searching an extremum is obtained:

$$\left.\begin{aligned} \frac{\partial J}{\partial q_k} &= 0 \\ \frac{\partial J}{\partial \mathring{q}_k} &= 0 \end{aligned}\right\} \tag{6}$$

by which a system of algebraic equations is formed.

ILLUSTRATIVE EXAMPLES

(1) *Mass spring and damper system*

The functional in accordance to Hamilton's extended principle would be:

$$J_1 = \frac{1}{2} m \int_{t_0}^{t_f} [\mathring{x}^2 - \omega^2 x^2 - (2\omega_1 \mathring{x})x]\,\mathrm{d}t. \tag{7}$$

Not using Hamilton's principle a second accurate functional would be [7]:

$$J_2 = \frac{1}{2} m \int_{t_0}^{t_f} e^{\omega_1 t}(\mathring{x}^2 - \omega^2 x^2)\mathrm{d}t. \tag{8}$$

Following eqns (5) and (6), with finite element descretization, a system of two algebraic equations for each element's node (k) would then be:

$$\left.\begin{aligned} \left(\frac{\partial J_1}{\partial x_k}\right)_E &= \Delta_E \int_0^1 [\dot{\mathbf{N}}^T \mathbf{x}^e(\dot{\mathbf{N}}^T - \omega_1 \mathbf{N}^T) \\ &\qquad - \omega^2 \mathbf{N}^T \mathbf{x}^e \mathbf{N}^T] \frac{\partial \mathbf{x}^e}{\partial x_k}\mathrm{d}\tau = 0 \\ \left(\frac{\partial J_1}{\partial \mathring{x}_k}\right)_E &= \Delta_E \int_0^1 [\dot{\mathbf{N}}^T \mathbf{x}^e(\dot{\mathbf{N}}^T - \omega_1 \mathbf{N}^T) \\ &\qquad - \omega^2 \mathbf{N}^T \mathbf{x}^e \mathbf{N}^T] \frac{\partial \mathbf{x}^e}{\partial \mathring{x}_k}\mathrm{d}\tau = 0 \end{aligned}\right\} \tag{9}$$

and for the second functional:

$$\left.\begin{aligned} \left(\frac{\partial J_2}{\partial x_k}\right)_E &= \Delta_E \int_0^1 e^{\omega_1(\Delta_E\tau + t_k)}[\dot{\mathbf{N}}^T \mathbf{x}^e \dot{\mathbf{N}}^T \\ &\qquad - \omega^2 \mathbf{N}^T \mathbf{x}^e \mathbf{N}^T) \frac{\partial \mathbf{x}^e}{\partial x_k}\mathrm{d}\tau = 0 \\ \left(\frac{\partial J_2}{\partial \mathring{x}_k}\right)_E &= \Delta_E \int_0^1 e^{\omega_1(\Delta_E\tau + t_k)}(\dot{\mathbf{N}}^T \mathbf{x}^e \dot{\mathbf{N}}^T \\ &\qquad - \omega^2 \mathbf{N}^T \mathbf{x}^e \mathbf{N}^T) \frac{\partial \mathbf{x}^e}{\partial \mathring{x}_k}\mathrm{d}\tau = 0. \end{aligned}\right\} \tag{10}$$

Letting $j = k$ and $j = k - 1$ intermittently, yields a system of linear algebraic equations that involve parameters about nodes $k-1$, k, $k+1$.

Integrating the recurrence system formed by Hamilton's extended principle, eqns (9), and letting:

$$\Delta \equiv \Delta_E = t_k - t_{k-1} = t_{k+1} - t_k;\ \Delta_E \ll 1$$

yields:

$$\left(\frac{\partial J_1}{\partial x_k}\right)_E = x_{k+1} - 2x_k + x_{k+1} = 0 \tag{11}$$

$$\left(\frac{\partial J_1}{\partial \dot{x}_k}\right)_E = x_{k+1} - x_{k-1} = 0. \qquad (12)$$

Multiplying eqn (12) by $\frac{1}{2}$ yields the Taylor expansion for x_{k+1} and x_{k-1} as functions of parameters about the node k.

$$\left.\begin{aligned} \dot{x}_k \Delta &= \tfrac{1}{2}(x_{k+1} - x_{k-1}) \\ \ddot{x}_k \Delta^2 &= x_{k-1} - 2x_k + x_{k+1} \end{aligned}\right\} \qquad (13)$$

Equations (13) are used as finite difference schemes that stand for the appropriate expressions in the equations of motion.

The following figures describe various examples that were solved using various combinations of boundary conditions. Curves that describe the difference from the closed form solution were observed to yield the following conclusions:

(a) *Solution based on Hamilton's principle.*

(1) Difference curves restrain faster to the line of zero difference, in as much as the damping coefficient grows up (Figs. 2c, d).

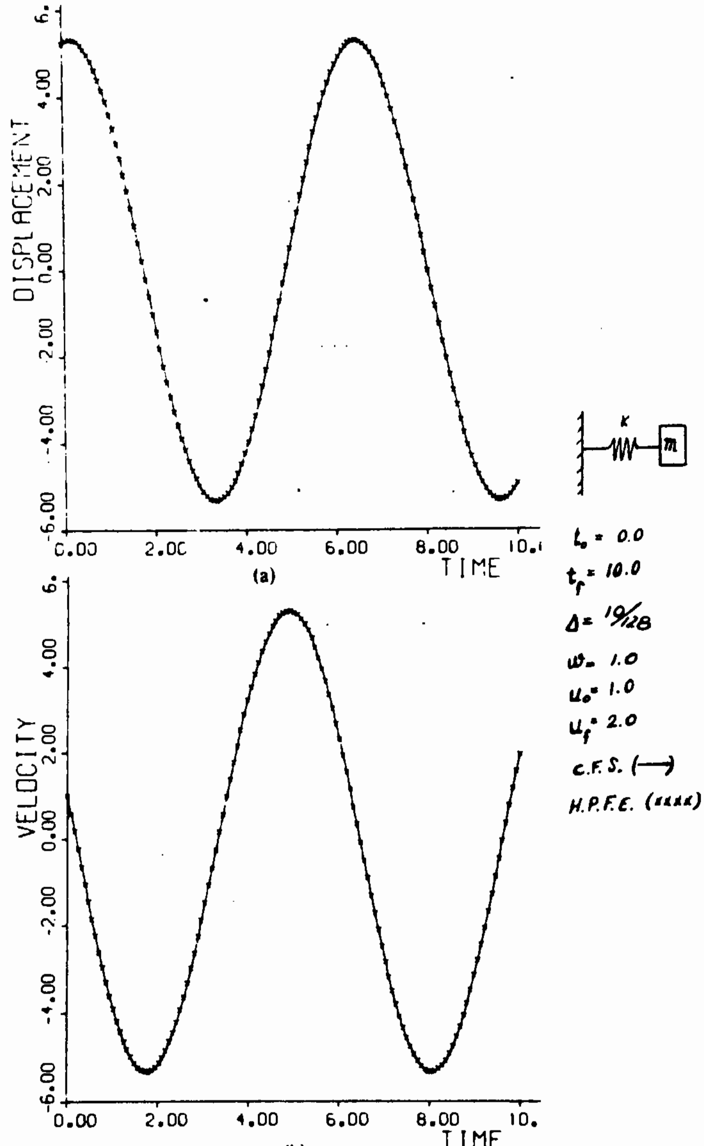
(2) When no damping is imposed, difference curves are limited within fixed limits (Figs. 1c, d, 4c, d).

(3) A very good correlation is noticed between the difference curves and the closed form solution curves (Figs. 1c, d, 4c, d).

Stability and convergence are due to the fact that the system of equations is solved simultaneously within the entire time interval.

(b) *Solution based on a functional such that all of its variables are varied.*

(1) Imposing damping yields a change in correlation between the difference and the solution curves. For high damping coefficients a great deviation, from the line of zero difference, occurs at the first time steps (Figs. 2e, f). Such a phenomenon is explained because of the expression $e^{\alpha t}$ in the process of numerical integration an accumulation of round-off errors is posed inasmuch as that exponent's power grows up.



(b)

Fig. 1.

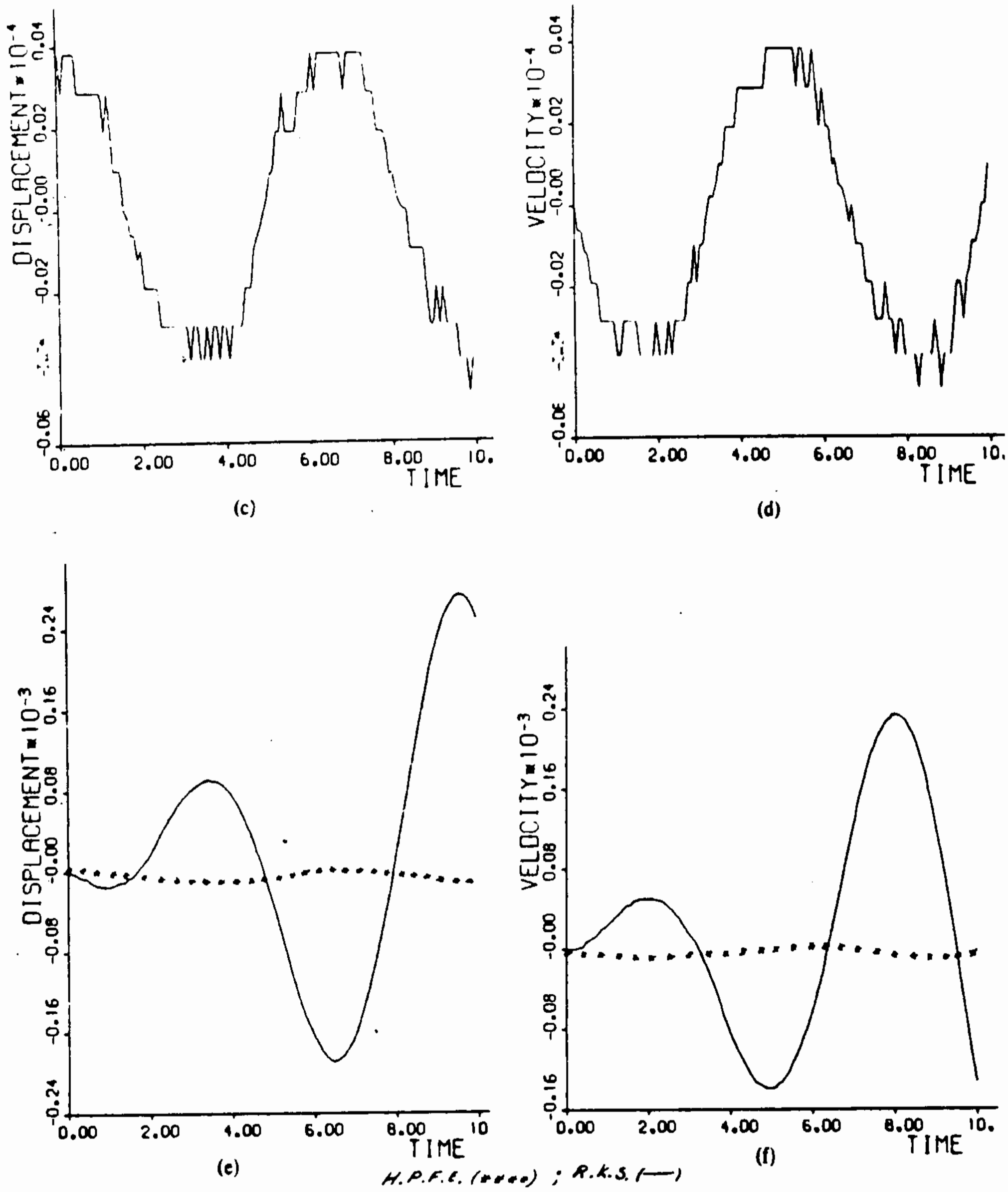


Fig. 1(a,b). Mass and spring-solution curves. (c,d). Mass and spring-difference curves. (e,f). Mass and spring-difference curves.

(c) *Solution based on Runge–Kutta's method.*

Initial values received from the closed form solution had been used as input data. The difference curve shows an instability nature and it reaches higher deviations, from the line of zero difference, compared to the difference curves of the present method (Figs. 1e, f).

(2) *Motion of a partical in a gravitation field, when drag proportional to any power of its velocity is present*

Hamilton's extended principle yields:

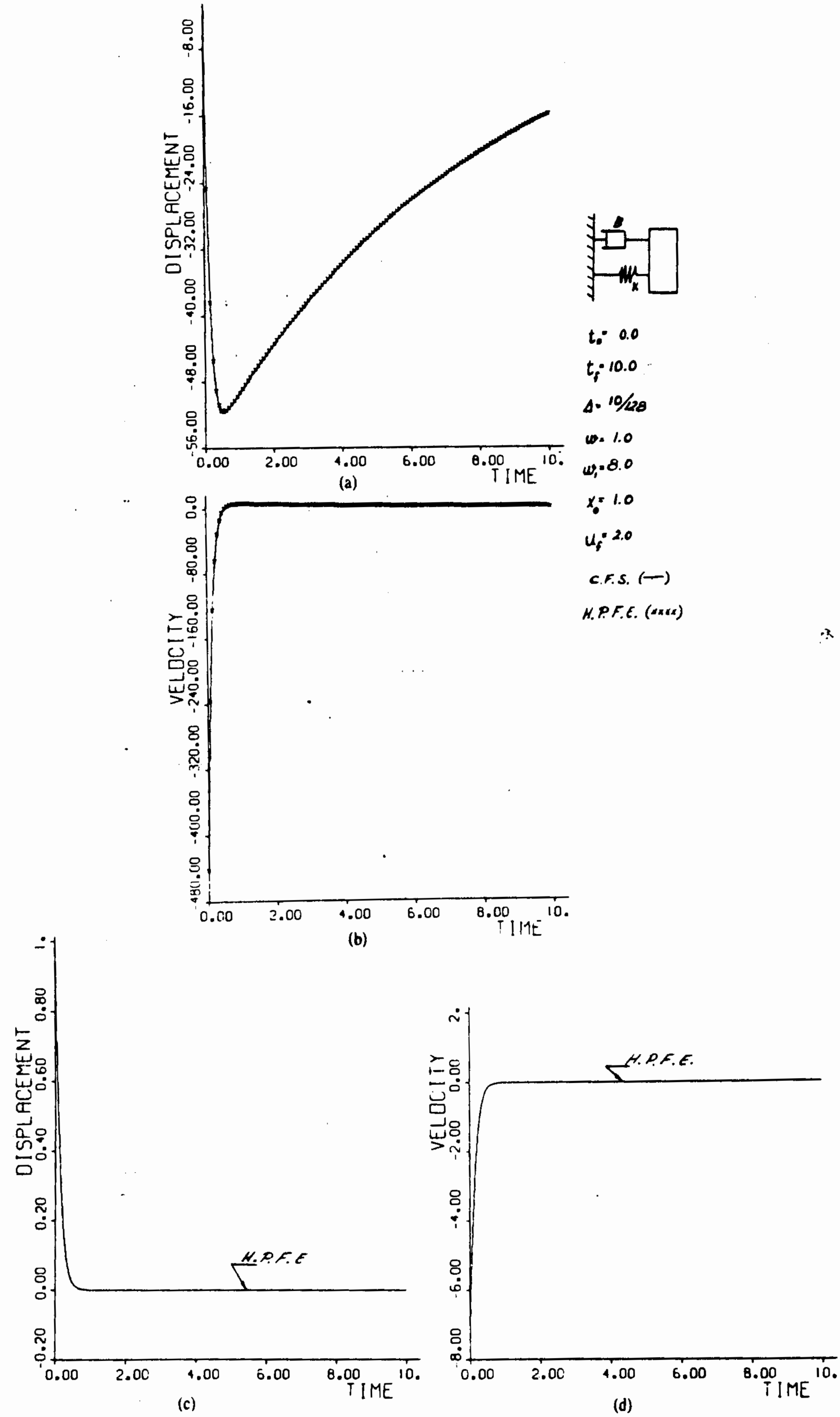
$$J = \tfrac{1}{2}m \int_{t_0}^{t_f} [\mathring{S}^2 - 2gy - 2\omega_1 \mathring{S}^n(\dot{x}x + \dot{y}y)]\,\mathrm{d}t. \qquad (14)$$

Inserting a finite element desecretization, eqn (2), for each directional coordinate, and in accordance with eqns (4)–(6), yields the following system of nonlinear equations defined about each element's node:

$$\left(\frac{\partial J}{\partial x_k}\right)_E = \Delta_E \int_0^1 \dot{\mathbf{N}}^T\mathbf{x}^e\{\dot{\mathbf{N}}^T - \omega_1[(\dot{\mathbf{N}}^T\mathbf{x}^e)^2 + (\dot{\mathbf{N}}^T\mathbf{y}^e)^2]^n\mathbf{N}^T\}\frac{\partial \mathbf{x}^e}{\partial x_k}\,\mathrm{d}\tau = 0 \qquad (15.1)$$

$$\left(\frac{\partial J}{\partial \dot{x}_k}\right)_E = \Delta_E \int_0^1 \dot{\mathbf{N}}^T\mathbf{x}^e\{\mathbf{N}^T - \omega_1[(\dot{\mathbf{N}}^T\mathbf{x}^e)^2 + (\dot{\mathbf{N}}^T\mathbf{y}^e)^2]^n\mathbf{N}^T\}\frac{\partial \mathbf{x}^e}{\partial \dot{x}_k}\,\mathrm{d}\tau = 0 \qquad (15.2)$$

$$\left(\frac{\partial J}{\partial y_k}\right)_E = \Delta_E \int_0^1 \dot{\mathbf{N}}^T\mathbf{y}^e\left\{\dot{\mathbf{N}}^T - \omega_1[(\dot{\mathbf{N}}^T\mathbf{x}^e)^2 + (\dot{\mathbf{N}}^T\mathbf{y}^e)^2]^n\mathbf{N}^T\right\}\frac{\partial \mathbf{y}^e}{\partial y_k}\,\mathrm{d}\tau \cancel{+} g\Delta_E \int_0^1 \mathbf{N}^T\frac{\partial \mathbf{y}^e}{\partial y_k}\,\mathrm{d}\tau = 0 \qquad (15.3)$$



15.1)

15.2)

15.3)

Fig. 2.

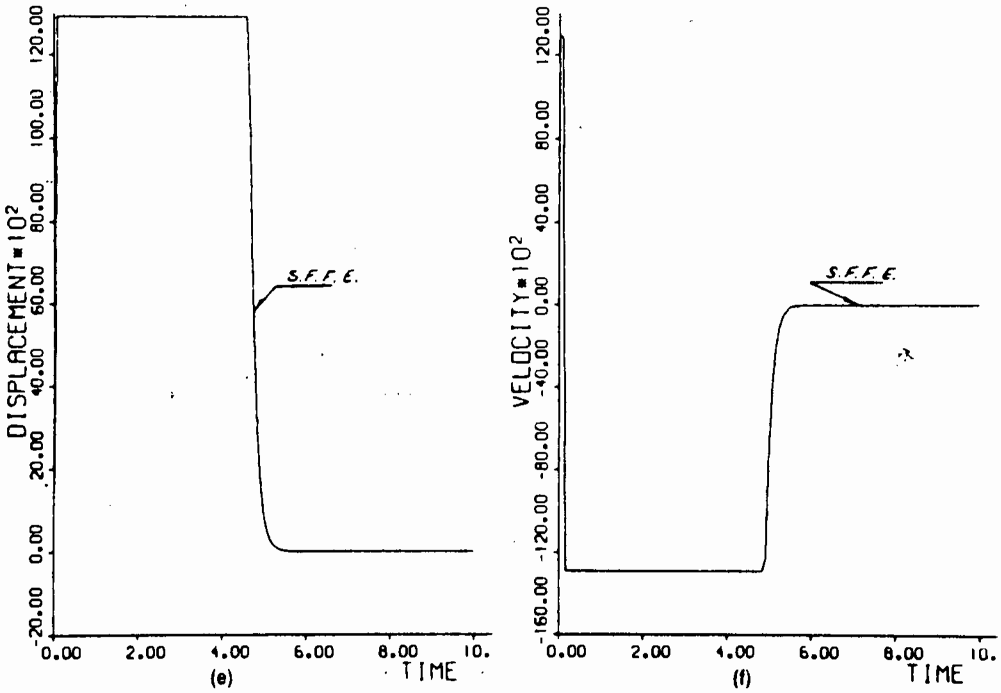


Fig. 2. (a, b) Mass, spring and damper-solutions curves. (c, d) Mass, spring and damper-difference curves. (e, f) Mass, spring and damper-difference curves.

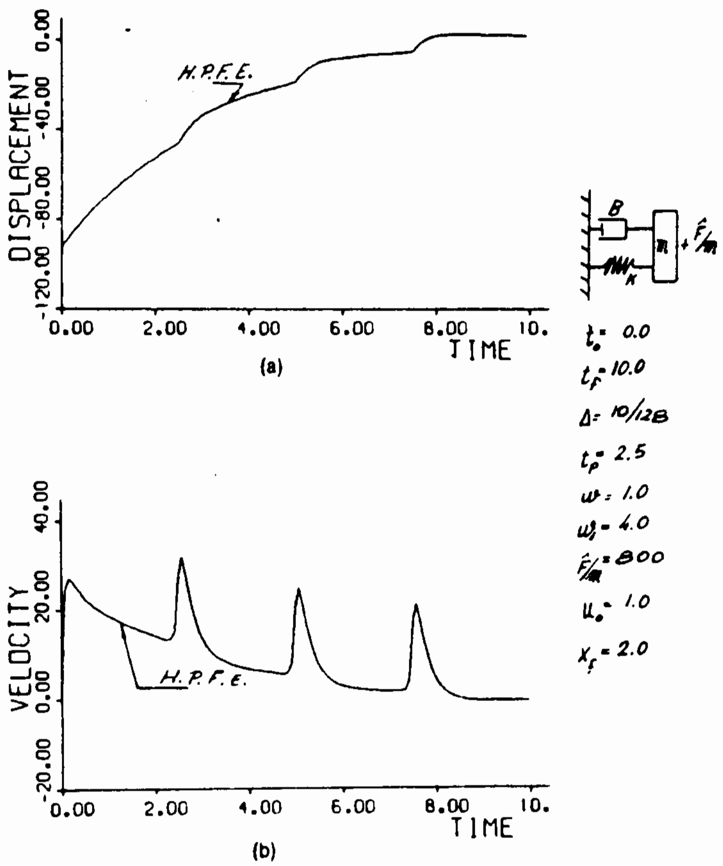


Fig. 3. Adding periodic impulse to a mass, spring and damper system.

$$\left(\frac{\partial J}{\partial \dot{y}_k}\right)_E = \Delta_E \int_0^1 \dot{\mathbf{N}}^T \mathbf{y}^e \{\dot{\mathbf{N}}^T - \omega_1 [(\dot{\mathbf{N}}^T \mathbf{x}^e)^2 + (\dot{\mathbf{N}}^T \mathbf{y}^e)^2]^n g \mathbf{N}^T\} \frac{\partial \mathbf{y}^e}{\partial \dot{y}_k} \, \mathrm{d}\tau \neq g\Delta_E \int_0^1 \mathbf{N}^T \frac{\partial \mathbf{y}^e}{\partial y_k} \, \mathrm{d}\tau = 0. \quad (15.4)$$

Letting $j = k$ and $j = k - 1$ intermittently, yields a system of nonlinear recurrence equations that involve parameters about nodes $k - 1$, k, $k + 1$.

Following are figures that illustrate some examples that had been solved. A comparison was made between solutions based on the present method and closed form solutions for various boundary conditions. For $\omega_1 = 0$ and $\bar{n} = 0$ (Figs. 4a, b, 5a, b), and a motion along a trajectory, by ascribing $\dot{x}/y \ll 1$ along the particle path (Figs. 6a, b).

(2) *Extension to initial value problems*:

Gurtin in his work presented a method of deriving a variational principle to linear differential field equations

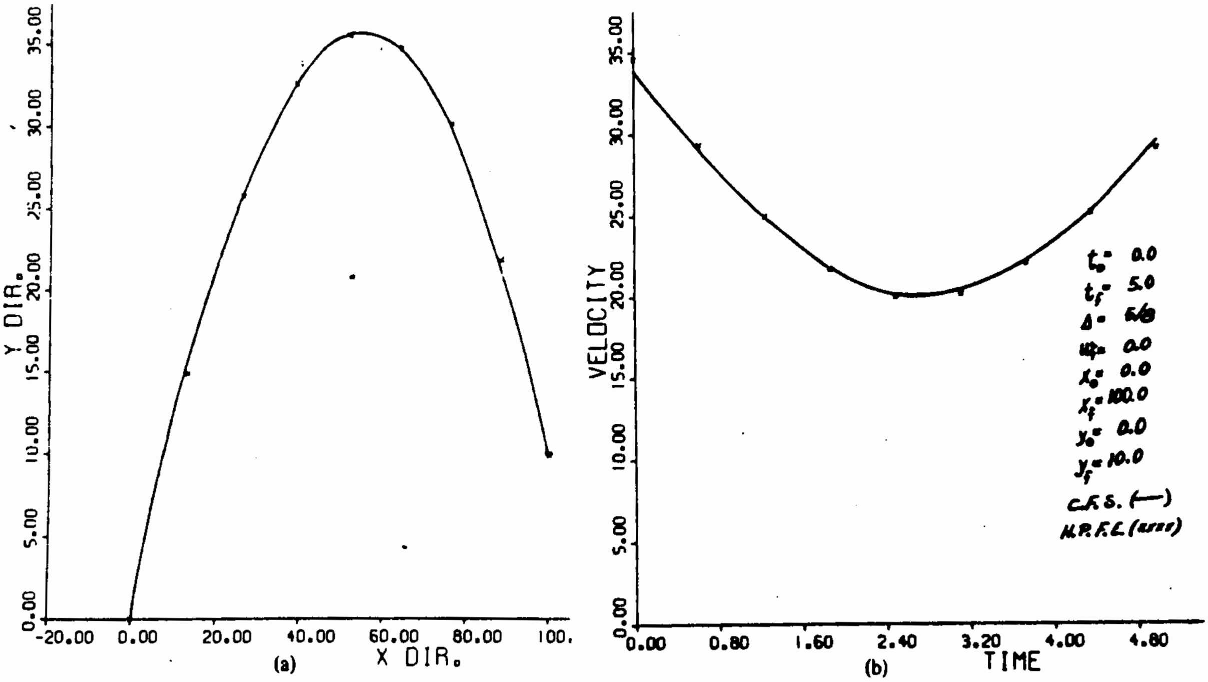


Fig. 4. Particle's motion without drag.

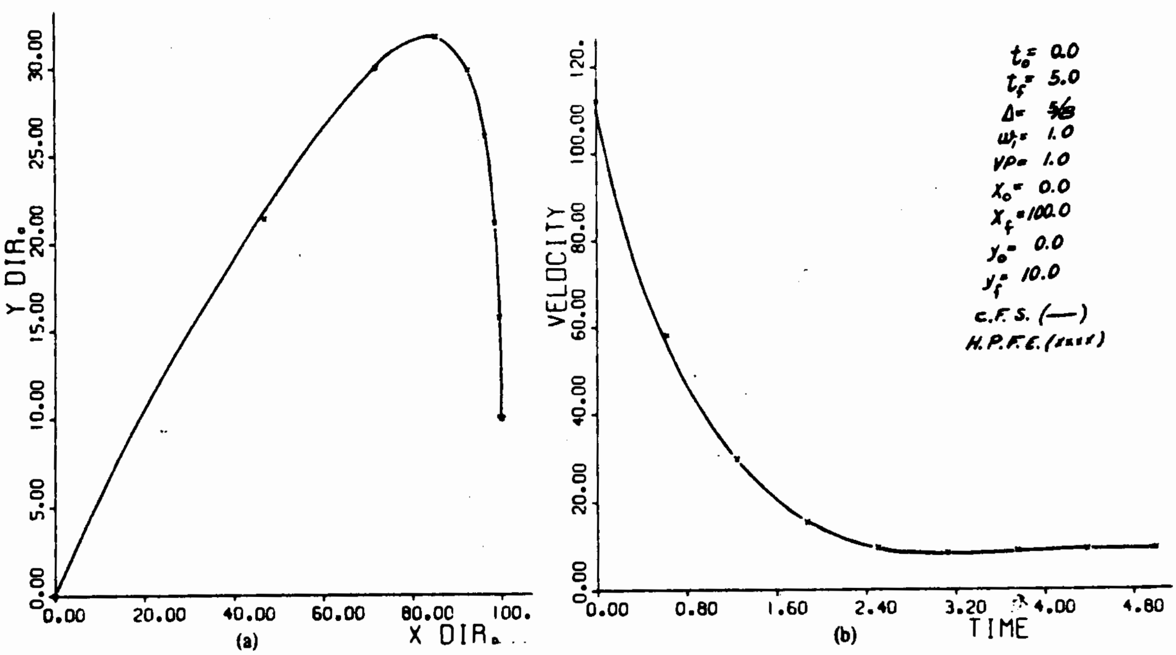
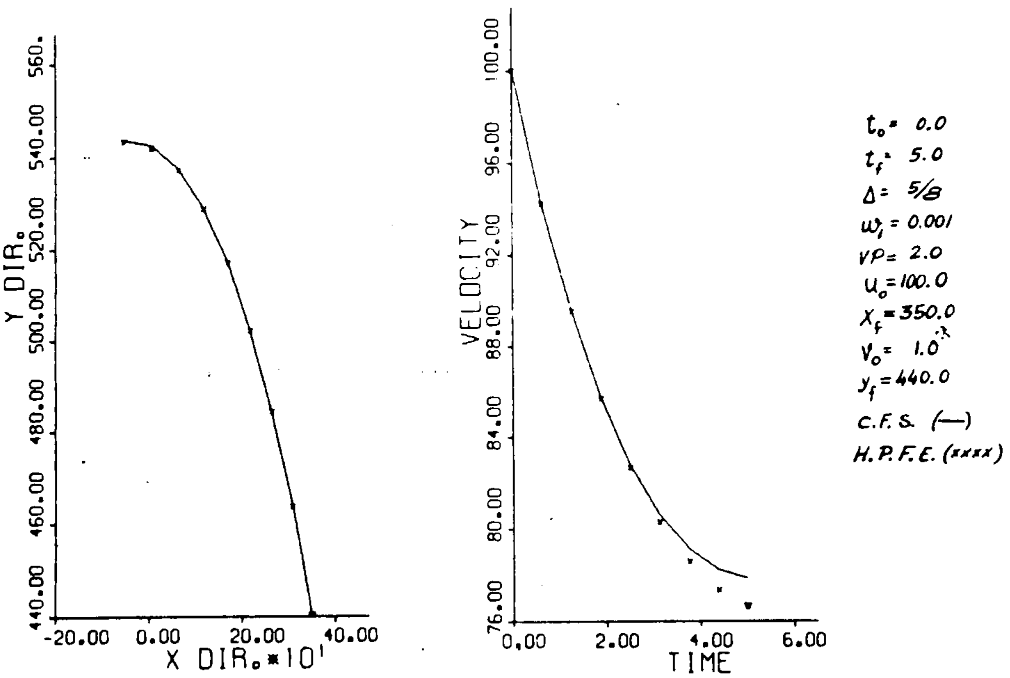


Fig. 5. Particle's motion for a first order velocity power.



(a) (b)

Fig. 6. Particle's motion along a flat path.

where initial and boundary conditions are incorporated explicitly in the functional. The nature of solution for an initial value problem is a step-by-step process in time. The present method can be applied to functionals in accordance with Gurtin's method to evaluate time variables after each time step. For means of illustration, let us consider a mass spring system. The motion equation would be:

$$m\ddot{x} = kx = 0 \tag{16}$$

where initial conditions are:

$$x(t = 0) = x_0$$
$$\dot{x}(t = 0) = u_0. \tag{17}$$

Following Gurtin's method a functional ascribed between limits of a time step would be

$$J = \int_{t_j}^{t_{j+1}} \{x * x + \omega^2 t * x * x - 2(x_0 + u_0 t * x)\, dt \tag{18}$$

where the convolution product $(*)$ is ascribed to the same time step limits.

Inserting the finite element desecretization, eqns (2)–(4), and following the procedure prescribed by eqns (5) and (6), yields a system of two algebraic equations about each element's node:

$$\left(\frac{\partial J}{\partial x_k}\right)_E = \Delta_E \int_0^1 \{[1 + \omega^2(\tau\Delta_E + t_k)*](\mathbf{N}^T\mathbf{x}^e) - [x_0 + u_0(\tau\Delta_E + t_k)]\}*\left(\mathbf{N}^T\frac{\partial \mathbf{x}^e}{\partial x_k}\right) d\tau = 0$$

$$\left(\frac{\partial J}{\partial \dot{x}_k}\right)_E = \Delta_E \int_0^1 \{[1 + \omega^2(\tau\Delta_E + t_k)*](N^T x^e) - [x_0 + u_0(\tau\Delta_E + t_k)]\}*\left(N^T\frac{\partial \mathbf{x}^e}{\partial \dot{x}_k}\right) d\tau = 0. \tag{19}$$

Letting $j = k$ yields a system of recurrence equations that involve parameters about nodes k, $k + 1$, which in fact indicates a time step to where all evaluations are valid.

CONCLUSIONS

The present method proved to give good correlation to a closed form solution when applied to Hamilton's extended principle. Applying the numerical analysis to Hamilton's principle, though it is a stationary principle, reached better numerical convergence and stability when compared to attacking the complete variational principle governing the same dynamical system.

For a solution which describes an added instantaneous phenomenon to the dynamical system, a simple process of adding the proper expressions to the system of equations was utilized. The present method may be applied to any functional which includes time as one of the independent variables. It can be worked out as a permanent routine for problems defined by governing functionals.

For boundary value problems, where a governing functional can be written, known routines such as Runge–Kutta are cumbersome to apply. Yet working out the problem with the present method is straightforward.

The present method can be applied to more than one independent variable. In such a case time will be one among other coordinates which define the element structure, extending all nodal points uniformly into the time dimension.

REFERENCES

1. U. Holzlohner, A finite element analysis for time dependent problems. *Int. J. Num. Meth. Engng* **8**, 55–69 (1974).
2. G. Warzee, Finite element and Laplace transform comparative solutions of transient heat conduction problems. *Comput. Structures* **4**, 979–991 (1974).

3. O. C. Zienkiewicz, *The Finite Element Method in Engineering Science*. McGraw Hill, New York (1971).
4. I. Fried, Finite element analysis of time dependent phenomena. *AIAA J.* 7(6), 1170–1173 (1969).
5. J. M. Vance and A. Sitchin, Numerical solution of dynamical systems by direct application of Hamilton's principle. *Int. J. Num. Meth. Engng* **4**, 207–216 (1972).
6. G. Leitmann, Some remarks on Hamilton's principle. *J. Appl. Mech.* 623–625 (1973).
7. D. R. Smith and C. V. Smith, Jr., When is Hamilton's principle an extremum principle? *AIAA J.* **12**(11), 1573–1576 (1974).
8. M. E. Gurtin, Variational principles for linear elastodynamics. *Arch. Rat. Mech. Anal.* **16**, 34–50 (1964).
9. M. E. Gurtin, Variational principles for linear initial value problems. *Quart. Appl. Math.* 252–256 (1964).
10. R. S. Sandhu and E. L. Wilson, Finite element analysis of seepage in elastic media. *J. Engng Mech. Div.* 641–652 (1969).
11. S. Sorek, Using the Finite-Element Technique for Solving Problems Formulated by Hamilton's Principle. M.Sc. thesis, Technion, Haifa, Israel (1975).
12. H. Goldstein, *Classical Mechanics*, pp. 30–56, Addison-Wesley, New York (1969).
13. W. Flügge, *Handbook of Engineering Mechanics* pp. 16-7–16-8, McGraw Hill, New York (1962).
14. E. L. Wilson and R. W. Clough, Dynamic response by step-by-step matrix analysis. *Symp. Use of Computers in Civil Engng*, Paper No. 45 (1962).